A Representation of Projective Space by the Points of a Line

by George Wellman Hess

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A REPRESENTATION OF PROJECTIVE SPACE BY THE POINTS OF A LINE.

1. It is the purpose of this paper to give a representation of the points, lines, and planes of a projective space of three dimensions by means of the points of a single line, $\bar{l}$, which we call the <u>fundamental line</u>. To this end we assume that certain projective transformations, or projectivities, which are of two kinds, non-degenerate and degenerate, are defined on $\bar{l}$ with the following properties:

I. <u>Non-degenerate Projectivities</u>:

1. A <u>non-degenerate projectivity</u> is a one-to-one correspondence between the points of $\bar{l}$ which has a unique one-to-one inverse.

2. A non-degenerate projectivity is completely determined when there are given any three distinct pairs of homologous points which belong to it. Here a homologous pair is used so as to include the case where the pair consists of coincident points. The foregoing statement then includes the determination of a non-degenerate projectivity by means of one double point (two double points) and two homologous pairs (one homologous pair) of distinct points.

3. If a projectivity leaves three distinct points of $\bar{l}$ invariant, it leaves every point of $\bar{l}$ invariant. This projectivity is called the *identical projectivity*.

4. If A,B,C,D are four points of $\bar{l}$, we have

$$ABCD \barwedge BADC.$$

5. If we have $MNAA' \barwedge MNBB'$, we also have

$$MNAB \barwedge MNA'B'.$$

6. A projectivity having one, and only one, double point is called *parabolic*.

7. A non-identical projectivity of period two is called an *involution*. Any projectivity giving $AA' \barwedge A'A$ for any single point A which is not a double point is an involution.

8. The class of all non-degenerate projectivities on $\bar{l}$ forms a group G.

II. *Degenerate Projectivities*. The only degenerate projectivities considered in this paper are of two classes.

1. *Class C'* consists of all degenerate projective transformations on $\bar{l}$ such that each projectivity of the class leaves the point M invariant and transforms another point of $\bar{l}$ successively into each point of $\bar{l}$ distinct from M.

2. *Class C"* consists of all degenerate projective

transformations on $\bar{l}$ such that each projectivity of the class leaves the point M invariant and successively transforms each point of $\bar{l}$ distinct from M into a single point of $\bar{l}$ distinct from M.

The class $C_m = [\pi]$ of all projectivities leaving the point M invariant is of fundamental importance in what follows. It consists of the classes C' and C" together with a subgroup G_m of G, which consists of all projectivities of G leaving the point M invariant. In what follows the letter π with or without subscripts or primes (π, π_1, π_i, π', ······) will always denote a projectivity of the class C_m.

III. Throws:

1. Two ordered pairs of points AB,CD constitute a throw, T(AB,CD). Two throws are said to be equal if they are projective; i.e. T(AB,CD) = T(A'B',C'D'), if, and only if,

$$AB,CD \;\bar{\wedge}\; A'B',C'D'$$

2. By I,4 we always have

$$T(AB,CD) = T(BA,DC) = T(CD,AB) = T(DC,BA).$$

3. If M and N are double points of a projectivity and AA' is any homologous pair ($A \neq M,N$), the throw T(MN,AA') is called the characteristic throw of the projectivity. This term is justified by II,5, since if BB' is any other homologous pair

($B \neq M,N$), we have $T(MN,AA') = T(MN,BB')$.

4. The characteristic throw of a parabolic projectivity is $T(MM,AA')$. Evidently the characteristic throws of any two parabolic projectivities are equal to each other and also equal to the throw $T(AB,CC)$, which is the characteristic throw of the identical projectivity. This throw is denoted T_{m}.

5. The characteristic throw of any projectivity of C' is $T(MA,AB)$, which will be denoted T_{∞}. The characteristic throw of any projectivity of C'' is $T(MB,AB)$, which will be denoted T_0.

6. Any projectivity of G is completely determined when its characteristic throw and two pairs of homologous points are given. Any degenerate projectivity of C_{m} is completely determined when its characteristic throw, the double point M, and another pair of homologous points are given.

In the foregoing statements nothing is assumed as to the number or nature of the points on $\bar{1}$ except that the existence of at least three distinct points is tacitly assumed.

2. One theorem concerning the projectivities of C_{III} which we shall use frequently is inserted here as an introductory lemma.

Introductory Lemma. (I.L.). If π_1 and π_2 are any two distinct projectivities of C_{III}, the necessary and sufficient condition that π_1 and π_2 shall have in common either a pair of distinct homologous points or a double point distinct from M is that π_1 and π_2 shall have different characteristic throws.

In the case where both π_1 and π_2 belong to G_{III}, if they have a common pair, say R_1 R_2, it is evident that the projectivity $\pi_2^{-1}\pi_1$ has R_1 for a double point, and conversely, if $\pi_2^{-1}\pi_1$ has a double point distinct from M, that π_1 and π_2 have a common pair. It follows from III,6 that if two <u>distinct</u> projectivities of G_{III}, ~~they~~ have their characteristic throws equal, they cannot have a common pair of distinct homologous points or a common double point distinct from M. Therefore if $T_1 = T_2$, $\pi_2^{-1}\pi_1$ is parabolic. If $T_1 \neq T_2$, it follows from III,4 that at least one of the projectivities π_1 and π_2, say π_2, has a double point distinct from M, say C. Then $\pi_2(MCA_1) = MCA_2$, where A_1 and A_2 form a pair of distinct homologous points of π_2. Let C' and A' be so defined that

$$\pi_1(MNC'A') = MNCA_2.$$

Then $\pi_2^{-1}\pi_1(MO'A') = MOA_1$. Let π' be the projectivity determined by T_2, the double point M, and the pair $O'O$. Then $\pi'^{-1}(A_2)$ will be distinct from A', say A''.

$$\pi_2^{-1}\pi'(MO'A'') = MOA_1.$$

$\pi_2^{-1}\pi_1$ and $\pi_2^{-1}\pi'$ are distinct projectivities of G_{π}, and therefore cannot both be parabolic, for if they were they would have in common the throw T_m as well as the double point M and the pair $O'O$. $\pi_2^{-1}\pi'$ is parabolic since $T_2 = T'$. Therefore $\pi_2^{-1}\pi_1$ has a double point distinct from M, and it follows that π_1 and π_2 have either a pair of distinct homologous points or a double point distinct from M in common.

The truth of the theorem has therefore been established for the case where neither π_1 nor π_2 is degenerate, and it is obvious if either π_1 or π_2 is degenerate.

Geometry of Planar Points.

3. *Definitions of Planar Points and Planar Lines.* If P_1 and P_2 are any two points of the fundamental line, each distinct from M, the ordered pair of points (P_1,P_2) constitutes an *ordinary planar point P*. P_1 (P_2) is called the *first* (*second*) *component* of the planar point P. The point $P = (P_1,P_2)$ is distinct from the point (P_2,P_1) whenever P_1 is distinct from P_2.

Any throw of points T(GH,IJ) on the fundamental line constitutes an *ideal planar point*. Any two ideal planar points coincide if, and only if, the throws of points on $\bar{l}$ which constitute them are equal.

If π is any projectivity of C_m, the totality of ordinary planar points consisting of pairs of homologous points of π together with the ideal planar point consisting of the characteristic throw of π constitutes an *ordinary planar line l*. The projectivity π which determines l is called the *projectivity of l*. In denoting a planar line and its projectivity the letters l and π will be given the same subscripts and primes.*

The totality of ideal planar points constitutes *the ideal planar line*.

The planar line whose projectivity is of ClassC' (Class C") is called a *special planar line of Class I (Class II)*.

*If a planar line is denoted by a, b, or other letter different from l, its projectivity will be denoted by π_a, π_b, etc.

4. <u>Theorems of Alignment for Planar Points</u>.

THEOREM I. If A and B are any two distinct planar points, A and B are on one, and only one, planar line.

This theorem follows immediately from I,2 except for the case where both A and B are ideal; and in this case it is evident that A and B are on the ideal planar line, and on no other line.

THEOREM II. If l_1 and l_2 are any two distinct planar lines, l_1 and l_2 are on one, and only one, planar point.

This follows immediately from I.L. for all cases except that in which either l_1 or l_2 is the ideal planar line, in which case the ideal point of the ordinary planar line is a common point, and is the only common point.

5. <u>Definitions</u>. <u>Planar Pencils</u>. The set of all planar lines l through a given planar point S forms a <u>pencil of planar lines</u> [l] on the point S, which is called the <u>center of the pencil</u> [l]. The set of all planar points P on a planar line l constitutes a <u>pencil of planar points</u> [P] on the line l, which is called the <u>axis of the pencil</u> [P]. The distinction should be made between the line l and the pencil [P] of points on l that in speaking of the pencil the points are thought of as separate points capable of entering into a one-to-one recipro-

cal correspondence with the elements of another planar pencil, which may be either a pencil of points on the same line l or on a line distinct from l or a pencil of planar lines $[l^i]$.

Planar Perspectivities. If $[P^i]$ and $[Q^i]$ are pencils of planar points on distinct planar lines l_p and l_q respectively having a one-to-one reciprocal correspondence established between them so that corresponding to each P^i there is a unique Q^i, and vice versa, and if there exists a planar point S not on l_p or l_q such that S is collinear with each P^i and its homologue Q^i, the pencil $[P^i]$ is said to be perspective from the center S with the pencil $[Q^i]$: $[P^i] \overset{S}{\underset{\wedge}{=}} [Q^i]$. This correspondence is called a planar perspectivity. There is evidently determined also a one-to-one reciprocal correspondence between $[P^i]$ and a pencil of lines $[l^i]$ on S in which each point P^i corresponds to the line through S on which it lies. This is also a planar perspectivity. With any planar point S not on l_p or l_q as center, it is always possible to determine pencils $[P^i]$ and $[Q^i]$ on lines l_p and l_q respectively so that $[P^i] \overset{S}{\underset{\wedge}{=}} [Q^i]$. But if $[P^i]$ and $[Q^i]$ are given with a correspondence already established between them, it is not always possible to find any point S such that $[P^i] \overset{S}{\underset{\wedge}{=}} [Q^i]$.

Fundamental Pencils. With the ordinary points of a planar pencil $[P^i]$ on an ordinary planar line l_p there are evidently associated in general two sets of points $[P_1^i]$ and $[P_2^i]$ of the fundamental line in which P_1^i and P_2^i are respectively the first and second components of P^i. Introducing the double point M of π_p, the projectivity of l_p, into each of the pencils $[P_1^i]$ and $[P_2^i]$ as the homologue of T_p, the ideal point of l_p, we have two projective sets of points $[P_1^i]$ and $[P_2^i]$ on $\bar{l}$, which we shall call the _first and second fundamental pencils_ respectively _of the planar pencil_ $[P^i]$. The introduction of M into the fundamental pencils is justified by the fact that the planar pencils $[P^i]$ and $[Q^i]$ on distinct planar lines l_p and l_q respectively will have their common point represented by M in each fundamental pencil if, and only if, l_p and l_q meet in an ideal point. The pair (M,M) must not be thought of as uniquely designating a point, except that since an ordinary line has only one ideal point, (M,M) uniquely designates a point of a pencil on any ordinary planar line which is known to be the axis of the pencil. In case the axis of the pencil $[P^i]$ is a special planar line of Class I (Class II) the first (second) fundamental pencil of $[P^i]$ degenerates into the double points of the projectivity of the axis. If M and N are the points into which such a pencil degenerates, the _pencil is said to be degenerate on N_.

Planar Projectivities. A *planar projectivity* between two pencils of points [P] and [Q] is a one-to-one reciprocal correspondence between [P] and [Q] established by means of a sequence of perspectivities finite in number.

$$[P] \,\overline{\wedge}\, [Q], \text{ if } [P] \overset{S_1}{\underset{\wedge}{=}} [P'] \overset{S_2}{\underset{\wedge}{=}} [P''] \cdots\cdots\cdots [Q'] \overset{S_n}{\underset{\wedge}{=}} [Q].$$

6. In this section will be presented a set of theorems leading to the fundamental theorem of projective geometry for planar points.

THEOREM III. If $[A^i]$ and $[B^i]$ are two pencils of planar points on distinct ordinary planar lines a and b respectively and $[A^i_j]$ $(j=1,2)$ and $[B^i_k]$ $(k=1,2)$ are non-degenerate fundamental pencils of $[A^i]$ and $[B^i]$ respectively, and if we have $[A^i] \overset{C}{\overline{\wedge}} [B^i]$, where C is any planar point not on a or b, then there exists a projectivity of G , say π, such that $\pi[A^i_j]=[B^i_k]$.

Case I. $C=(C_1,C_2)$ is an ordinary planar point. As a first step we will show that the theorem holds for Case I if b is a special line aof Class I whose first fundamental pencil is degenerate on B_1 and a is not a special line. Let C_1^i and B^i_a be points of $\bar{1}$ such that $\pi_a(C_1^i\ B^i_a) = C_2 B^i_2$.

$$\text{Then} \quad \pi_a(MC_1^i\ A^i_1\ B^i_a) = MC_2 A^i_2\ B^i_2\ ,$$

$$\pi_i(MC_1\ A^i_1\ B_1) = MC_2 A^i_2\ B^i_2\ ,$$

$$\text{and} \quad \pi_i^{-1}\pi_a(MC_1^i\ A^i_1\ B^i_a) = MC_1 A^i_1\ B_1.$$

From I,4 we have $MC_1^i\ A^i_1\ B^i_a \,\overline{\wedge}\, A^i_1 B_1 MC_1$.

It follows from I,7 that the preceding projectivity is an involution and hence $MC_1' B_a^i B_1 \barwedge A_1^i B_1 C_1 C_1'$, from which by I, 4

$$MC_1' B_a^i B_1 \barwedge C_1 C_1' A_1^i B_1 .$$

Therefore $[B_a^i] \barwedge [A_1^i]$, and since $\pi_a[A_1^i] = [A_2^i]$ and $\pi_a[B_a^i] = [B_2^i]$, it follows that $[A_1^i] \barwedge [A_2^i] \barwedge [B_2^i]$.

A similar course of reasoning leads to similar result if $[A_1^i]$, $[A_2^i]$, or $[B_2^i]$ is degenerate instead of $[B_1^i]$.

If both a and b are special lines, it is only necessary to consider the pencil $[E^i] \overset{O}{=} [l^i]$, where $[E^i]$ has for its axis a non-special ordinary planar line of which O is not a point. Then each of the fundamental pencils of $[E^i]$ is projective with the non-degenerate fundamental pencil on each of the pencils $[A^i]$ and $[B^i]$. Therefore the non-degenerate fundamental pencils of $[A^i]$ and $[B^i]$ are projective with each other.

If neither a nor b is a special line, a somewhat similar course of reasoning is employed. The auxiliary planar pencil has for its axis a special line.

Case II. Let O be an ideal planar point, say T.

Let A^i be the point common to a and l^i, and $N^i, = (N^i, N^i)$, the double point of π^i, the projectivity of l^i. If R is any point of a whose components are distinct, it determines with T a line r. Let the double points of π_a and π_r be Q and K respectively. Then we have the following projectivities:

$$\pi_a \qquad M\,Q\,A^i_1\,R_1 \;\overline{\wedge}\; M\,Q\,A^i_2\,R_2 \,,$$

$$\pi^i \qquad M\,N^i\,A^i_1 \;\overline{\wedge}\; M\,N^i\,A^i_2 \,,$$

$$\pi_r \qquad M\,K\,R_1 \;\overline{\wedge}\; M\,K\,R_2 \,,$$

$$\pi_x \qquad M\,N^i\,A^i_1\,A^i_2 \;\overline{\wedge}\; M\,K\,R_1\,R_2 \,, \text{ from } \pi_r \text{ and } \pi^i$$

$$\pi'_a \qquad M\,Q\,A^i_1\,A^i_2 \;\overline{\wedge}\; M\,Q\,R_1\,R_2 \,, \text{ from } \pi_a \text{ by I,5,}$$

π_x and π'_a are the same projectivity by I,2 , and therefore

$$M\,Q\,N^i\,A^i_2 \;\overline{\wedge}\; M\,Q\,K\,R_1 \,, \text{ and by I,5}$$

$$M\,Q\,N^i\,K \;\overline{\wedge}\; M\,Q\,A^i_1\,R_1 \,.$$

Therefore $[A^i_1] \;\overline{\wedge}\; [N^i]$, which is both first and second fundamental pencil of that planar pencil perspective with $[l^i]$ on the planar line whose projectivity is the identical projectivity, which line is called the identitical line, or for brevity m. Therefore the theorem holds if either a or b coincides with m. If $a \neq m$ and $b \neq m$, the planar pencil on m which is perspective with $[l^i]$ can be introduced as an auxiliary pencil, and the theorem is then seen to hold for this case also. The proof just given for Case II does not hold, however, if T_α, T_0, or T_m is taken as the center of the perspectivity. If T_α or T_0 is used the truth of the theorem is obvious. The proof for the case where T_m is the center of the pencil $[l^i]$ is still outstanding.

THEOREM IV. If a pencil of points $[T^i]$ on the ideal planar line is projected by means of two ordinary planar points P and Q as centers into the pencils $[A^i]$ and $[B^i]$ on the ordinary planar lines a and b respectively, it follows that there is a projectivity which transforms a non-degenerate fundamental pencil of $[A^i]$ into a non-degenerate fundamental pencil of $[B^i]$.

Case I. $P_1 \neq P_2$, $Q_1 \neq Q_2$. Let $[l^i]$ and $[l^{ii}]$ be the pencils of lines on P and Q respectively, and let N^i and N^{ii} be the double points of π^i and π^{ii} respectively.

Then $\pi^i(M\ N^i\ P_1) = MN^i\ P_2$ and $\pi^{ii}(M\ N^{ii}\ Q_1) = M\ N^{ii}\ Q_2$. Since l^i and l^{ii} are concurrent in an ideal point, π^i and π^{ii} have equal characteristic throws, and

$$M\ N^i\ P_1\ P_2 \overline{\wedge} M\ N^{ii}\ Q_1 Q_2.$$

$\therefore [N^i] \overline{\wedge} [N^{ii}]$, where $[N^i]$ and $[N^{ii}]$ are the double fundamental pencils, so to speak, of planar pencils $[N^i]$ and $[N^{ii}]$ on the identical line m. By Theorem III, Case I, $[A^i] \overline{\wedge} [N^i]$ and $[B^i] \overline{\wedge} [N^{ii}]$. $\therefore [A^i] \overline{\wedge} [B^i]$.

Case II. Either P or Q is on, or both are on, m. The proof for this case is still outstanding.

THEOREM V. If [P] and [Q] are pencils of planar points on the ordinary planar lines p and q respectively such that $[P] \,\overline{\wedge}\, [Q]$, then there is a non-degenerate fundamental pencil of [P] which is projective with a non-degenerate fundamental pencil of [Q].

In the sequence of perspectivities which determines the planar projectivity $[P] \,\overline{\wedge}\, [Q]$, each planar pencil which has an ordinary planar line for its axis has at least one non-degenerate fundamental pencil, which by Theorem III or Theorem IV is projective with a non-degenerate fundamental pencil of the next planar pencil of the sequence which has an ordinary planar line for its axis. For two successive planar pencils of the sequence cannot have the ideal line as axis. Thus it is possible to establish a sequence of projectivities of G between successive fundamental pencils leading from a non-degenerate fundamental pencil of [P] to a non-degenerate fundamental pencil of [Q].

THEOREM VI. *The fundamental theorem of projectivity for planar points.* If A, B, C, and D are any four distinct planar points of a planar line l, and if $A\ B\ C\ D \,\overline{\wedge}\, A\ B\ C\ D'$, it follows that $D = D'$.

Case I. l is an ordinary planar line. By Theorem V the planar projectivity $A\ B\ C\ D - A\ B\ C\ D'$ determines on $\bar{l}$ at least one projectivity of G, $A_i\ B_i\ C_i\ D_i \,\overline{\wedge}\, A_j\ B_j\ C_j\ D_j'$ $(i=1,2)$ $(j=1,2)$.

It is also possible to take $i=j$, since if l is a special line, i is necessarily equal to j, and otherwise $[P_1] - [P_2]$. Hence by I,3 ,$D_i = D_i$, and therefore $D = D'$

Case II. l is the ideal line. Let A, B, C, D, and D' be projected from a point not on l into $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, and $\bar{D}'$ respectively. Then in the planar projectivity, $\bar{A}\ \bar{B}\ \bar{C}\ \bar{D} \overline{\wedge}\ \bar{A}\ \bar{B}\ \bar{C}\ \bar{D}'$, $\bar{D} = \bar{D}'$ by Case I, and therefore $D = D'$.

Having thus established the Fundamental Theorem for planar points, all theorems of projective geometry in a plane may be derived. In particular it is now possible to prove the Theorem of Desargues concerning perspective triangles which, as is well known, is not derivable from the theorems of alignment in the plane alone (although it can be proved from the theorems of alignment in space without the Fundamental Theorem). We will state it for reference as follows:

<u>Theorem of Desargues</u>. If two triangles in the same plane are perspective from a point, the three pairs of homologous sides meet in collinear points.

GEOMETRY OF SPATIAL POINTS.

7. Definitions of spatial points and ordinary spatial lines.

An *ordinary spatial point* consists of an ordered triple of points on the fundamental line $\bar{l}$, each point of the triple being distinct from the point M. $P = (P_1,P_2,P_3)$, $P_i \neq M$, $i = 1,2,3$.

An *ideal spatial point* T consists of an ordered triple of throws of points (T_1,T_2,T_3) on the fundamental line $\bar{l}$ such that there are three projectivities of C_m, π_1,π_2,π_3 of which T_1,T_2,T_3 are the respective characteristic throws, and that the product $\pi_3\pi_2\pi_1$ leaves invariant every point of $\bar{l}$ upon which π_1 operates. If T and T' are two ideal spatial points such that two components of T are respectively equal to the two corresponding components of T', the remaining component of T equals the remaining component of T' unless both T and T' have T_0 and T_∞ as components, in which case the remaining component of either ideal point is undetermined by the components given.

If π',π'', π''' are three projectivities of C_m such that at least one of the three products $\pi'''\pi''\pi'$, $\pi''\pi'\pi'''$, $\pi'\pi'''\pi''$ leaves $\bar{l}$ pointwise invariant, the ordered triple of projectivities (π',π'',π''') determines an *ordinary spatial line* l, which consists of the set of all ordinary spatial points $[X^i]$ such that $\pi'(X^i_1) = X^i_2$, $\pi''(X^i_2) = X^i_3$, and $\pi'''(X^i_3) = X^i_1$, together with the ideal spatial point $T = (T_1,T_2,T_3)$, where T_1,T_2,T_3 are the characteristic throws of π', π'', and π''' respectively. The projectivities π', π'', and π''' are called the first, second, and third projectivities of l respectively.

The set of all spatial points on any spatial line l constitutes a pencil of spatial points [P]. The line l is called the axis of the pencil. If π', π'', and π''' are all non-degenerate, corresponding to the pencil [P], there are three projective sets of points on $\bar{l}$ $[P_1]$, $[P_2]$, and $[P_3]$, which are called the first, second, and third fundamental pencils of [P] respectively.

If π' is of Class C' degenerate on a point P of $\bar{l}$, it is evident either that π''' must be of Class C" degenerate on P and π'' of G_m, or π'' must be of Class C" degenerate on a point, say Q, of $\bar{l}$ and π''' of G_m, transforming Q into P. We shall set aside to a certain extent the distinction between a pencil of spatial points and its axis and speak of a fundamental pencil of [P] as a fundamental pencil of the axis l. Where this is done, it will be justified by the fact that the properties of an axis as determined by its projectivities determine the degeneracy or non-degeneracy of the fundamental pencils of the spatial pencils on that axis.

An ordinary spatial line whose first (second, third) fundamental pencil is degenerate on N, and no other of whose fundamental pencils is degenerate, is a special ordinary spatial line of degenerate on $\mathfrak{N}$.

Type I, Class I (II,III). Such a line is denoted l of I_2 (I_2,I_3). on $\mathfrak{N}$
An ordinary spatial line which has two of its fundamental pencils degenerate is a special ordinary spatial line of Type II, Class I (II,III) if its first (second, third) fundamental pencil is non-degenerate. Such a line is denoted l of II_1 (II_2, II_3)

8. In this section is given a single theorem of alignment for spatial points which is needed as a prerequisite for certain definitions.

THEOREM VII. If A and B are any two distinct spatial points which are not both ideal, A and B are on one, and only one, ordinary spatial line.

Case I. Let both A and B be ordinary spatial points.

There is in general a single projectivity of C_m which transforms A_1B_1 into A_2B_2, asingle projectivity of C_m which transforms A_2B_2 into A_3B_3, and a single projectivity of C_m which transforms A_3B_3 into A_1B_1. Let these three projectivities be denoted π',π'', and π''' respectively. Then they determine the line AB of which they are the first, second, and third projectivities respectively. If $A_1 = B_1$ and $B1 = B_2$, the proof just given breaks down; but the truth of the theorem is then obvious, the line AB being a line of II_3. The truth of the theorem is also obvious when AB is of I_1, I_2, I_3, II_1, or II_2.

Case II. Let A be an ordinary spatial point and B be an ideal spatial point T.

By III,6, T_1, M, and the pair A_1A_2 determine a projectivity of

of C_m, say π'. T_2, M, " " " A_2A_3 " " "

" " " π'' T_3, M, " " " A_3A_1 " " "

A " " π'''. Then(π',π'',π'') determines a unique ordinary spatial line containing A and B (=T).

9. Definitions of spatial planes and ideal spatial lines.

If l is any ordinary spatial line and A any ordinary spatial point not on l, the totality of spatial lines determined by A and X, X being a variable point representing any point of l, constitutes an ordinary spatial plane, which will be denoted α. Since a spatial line consists of spatial points, the spatial plane α may also be considered as a class of spatial points of which each spatial line on A is a subclass.

The totality of all ideal spatial points form the ideal spatial plane.

The totality of all ideal spatial points on an ordinary spatial plane constitutes an ideal spatial line, which will be denoted λ_α, or λ, The ideal spatial line λ_α of the plane α is obviously a subclass of the class of points α.

If l_n is a special ordinary spatial line of I_1 (I_2, I_3) on N, and A_n an ordinary spatial point with N for its first (second, third) component, the spatial plane α_n determined by l_n and A_n is a <u>special spatial plane of Type I, Class I (II,III) on N</u>, which is denoted α of I_1 (I_2, I_3).

The ideal line of a special spatial plane α of I_1 (I_2, I_3) is a <u>special ideal spatial line of Type I_1 (I_2, I_3)</u>, which is denoted λ of I_1 (I_2, I_3).

10. THEOREM VIII. If P_n is an ordinary spatial point of α of I_1 (I_2, I_3) on N, the first (second, third) component of P_n is N.

This follows at once from the fact that any line of the pencil defining α_n contains two points having N for a common first (second, third) component.

THEOREM IX. If P_n is an ordinary spatial point having N for its first (second, third) component and $\alpha_n = \alpha$ of I_1 (I_2, I_3) P_n is a point of α_n.

Let α_n of I_1 be determined by A_n and l_n, where $A_n = (N, A_2, A_3)$, and l_n is determined by B_n and C_n, $B_n = (N, B_2, B_3)$, $C_n = (N, C_2, C_3)$. and let $P_n = (N, P_2, P_3)$. Let π_1', π_2'', π_3'' be the projectivities of l_n and let π_1', π_2'', π_2'' be the projectivities of the line l_2 $(l_2 = A_n P_n)$.

Then π_1'' and π_2'' have either a common pair of homologous points in common, say R_2 R_3, or a common characteristic throw T_r. Suppose the former is true. then $R_n = (n, R_2, R_3)$ is an ordinary spatial point common to l_n and l_2. Hence l_2 is a line of α_n and therefore P_n is a point of α_n. If π_n^* and π_2'' have a common characteristic throw, l_n and l_2 have an ideal spatial point in common, and the proof follows as in the other case.

Cor. I The totality of ordinary spatial points each of which has the same first (second, third)component constitute the totality of ordinary spatial points of a special spatial plane of I_1 (I_2,I_3).

Cor. II. Any special spatial plane of Type I is determined by any three of its points which are non-collinear.

Cor. III. Any special ideal spatial line of I_1 (I_2,I_3) consists of the totality of ideal spatial points of the form $(T_\infty, T_x,$ (T_∞,T_x,T_o) [(T_o,T_∞,T_x) or (T_x,T_o,T_∞)], where T_x is a variable throw taking all possible values, together with a special point (T_∞,T_o,T_m) [(T_m,T_∞,T_o) or (T_o,T_m,T_∞)] and a special point (T_m,T_∞,T_o) [(T_o,T_m,T_∞)or (T_∞,T_o,T_m)]

Cor. IV. Any ideal spatial line of I_1 KI_2,I_3L is uniquely determined by two of its points, and there is,therefore, one, and only one, λ of I_1 (λ of I_2, λ of I_3).

Cor. V. If α_n and α_o are each of I_1 (I_2,I_3) on N and O respectively,N$\neq$O, α_n and α_o have λ of I_1 (λ of I_2, λ of I_3) in and no common spatial point not on that line.

THEOREM X. If B and C are distinct points of the ordinary spatial line l and A is any ordinary spatial point not on l, and if the first (second, third) projectivities of any two of the three lines l, AB, AC have a common characteristic throw T_K , $T_K \neq T_0, T_\infty$, every ordinary line of the spatial plane determined by A and l has T_K for its first (second, third) projectivity.

Let $AB = l_1$ and $AC = l_2$. Then we have

π : $MNB_1C_1 \overline{\wedge} MNB_2C_2$ $\quad \therefore MNB_1B_2 \overline{\wedge} MNC_1C_2$

π_1 $MOA_1B_1 \overline{\wedge} MOA_2B_2$ $\quad \therefore MOA_1A_2 \overline{\wedge} MOB_1B_2$.

Now if $T = T_1$, $MNB_1B_2 \overline{\wedge} MCB_1B_2$, and N=C. and $\pi=\pi_1$.

Let D and E be any ordinary spatial points on l and l_1 respectively, Then from π : $MND_1 \overline{\wedge} MND_2$

" π_1: $MNE_1 \overline{\wedge} MNE_2$

$\therefore$ $MND_1D_2 \overline{\wedge} MNE_1E_2$, and $MNDD_1E_1 \overline{\wedge} MND_2E_2$,

This projectivity which is the first projectivity of DE has its characteristic throw and two double points in common with π and π_1. And the line l_2 is a line DE.

Cor. I. $\pi=\pi_1=\pi_2$ in the proof just given.

Cor. II. The ideal line of such a plane consists of a set of ideal spatial points having T_k as a common first (second, third) component.

Definitions. An ideal spatial line in which the first (second, third) components of each of its points is the same throw, say T_β, is an *ideal spatial line of Type II, Class I (II, III) on T_β*, denoted λ of II_1 (II_2, II_3) on T_β. An ordinary spatial plane containing such a line is a *special plane of Type II, Class I (II, III) on T_β*, denoted α of II_1 (II_2, II_3) on T_β. Any ordinary spatial plane which is not of Type I or Type II is of *general type* which may be denoted α^g.

THEOREM XI. If λ_β is of II_1 (II_2, II_3) on T_β, and T^β is any ideal spatial point having T_β for its first (second, third) component, then T^β is a point of λ_β.

Let l (=BC) and A determine α of II_1.

If (π', π'', π''') are the projectivities of l, $T^\beta = (T_\beta, T_\gamma, T_\delta)$, two cases may arise, $T_\gamma = T''$ and $T_\gamma \neq T''$. In the former case, $T\ (=T', T'', T''') = T^\beta$, which therefore lies on l and is a point of λ_β.

If $T_\gamma \neq T''$, let π_γ be the projectivity determined by T_γ, the double point M, and the pair $A_2\ A_3$. Then by I. L., π_γ and π'' have a common pair of homologous points, say R_2R_3. If $\pi'^{-1}(R_2) = R_1$, $R = (R_1, R_2, R_3)$ is a point of l. $MA_1R_1 \barwedge MA_2R_2$ defines a projectivity, say π_β, which has $T_\beta(=T')$ for its characteristic throw. For A and R both lie in α of II_1 on T_β.

$$\pi_\gamma \pi_\beta (MA_1R_1) = MA_3R_3.$$

T_δ is determined uniquely as the characteristic throw of π_δ, Then $T^{\beta} = (T_{\beta}, T_{\gamma}, T_{\delta})$ is the ideal point of the line AR, and therefore is a point of λ_{β}.

Cor. I. There is one , and only one, λ of II_1 (II_2, II_3) on T_{β}.

Cor. II. The line λ of II_1 (II_2, II_3) is determined by any two distinct points on it.

Cor. III. Any plane α of II_1 (II_2, II_3) is determined by any three of its points which are non-collinear.

Cor. IV. Any two distinct lines of a given plane α of II_1 $II_2, II_3)$ have one, and only one, point in common.

As an immediate consequence of Theorem IX, Cor.IV, Theorem XI, Cor. II, and the definitions we have the following Theorem.

THEOREM XII. If T^{ρ} and T^{σ} are any two ideal points of an α of general type, T^{ρ} and T^{σ} do not have a common first (second, or third component.

Cor. I. Any ideal point of an α^g is determined uniquely by its first, second, or third component and α^g.

THEOREM XIII. In any given ordinary spatial plane of general type α^g there is one, and only one, ideal spatial point which has a given throw T_{β} for its first (second, third) component.

Let α^g be determined by A and l. It is only necessary to prove that if the ideal point of l does not have its first (second, third) component equal to T_{β}, some other ideal point of α^g does. Let π_{β} be the projectivity determined by T_{β}, the double point M, and the pair of homologous points A_1A_2. Then π_β and π' have a common pair, say R_1R_2, which determines a point, $R=(R_1,R_2,R_3)$, of 1. The ideal point of the line AR is an ideal point of α^g with the given component, and it is the only such point.

11. Theorem XIV If A, B, and C are any three non-collinear spatial points not all of which are ideal, and X and Y are any two points on AB and AC respectively, there is one, and only one, spatial point collinear with X and Y and with B and C.

Case I. A,B,and C are all ordinary.

Let the lines AB, AC, BC, and XY be denoted l_1, l_2, l_3, and l_4 respectively.

$\pi_1'(MA_1B_1X_1) = MA_2B_2X_2$. $\qquad \pi_1''(MA_2B_2X_2) = MA_3B_3X_3$.

$\pi_2'(MA_1C_1Y_1) = MA_2C_2Y_2$. $\qquad \pi_2''(MA_2C_2Y_2) = MA_3C_3Y_3$.

$\pi_3'(MB_1C_1R_1) = MB_2C_2R_2$. $\qquad \pi_3''(MB_2C_2S_2) = MB_3C_3S_3$.

$\pi_4'(MX_1Y_1R_1) = MX_2Y_2R_2$. $\qquad \pi_4''(MX_2Y_2S_2) = MX_3Y_3S_3$.

R_1 and R_2 denote any pair of homologous points, each distinct from M, which is common to π_3' and π_4', S_2 and S_3 any such pair common to π_3'' and π_4''.

The former pair exists if, and only if, $T_3' \neq T_4'$.

" latter " " " " " " $T_3'' \neq T_4''$.

To prove the theorem for this case it is only necessary to show that if $T_3' = T_4'$, $T_3'' = T_4''$, and if $T_3' \neq T_4'$, $R_2=S_2$.

Let us consider first the case where $T_3' \neq T_4'$ and $T_3'' \neq T_4''$.

Each projectivity of a spatial line determines a planar line, the ordinary points of which consist of the homologous point pairs of the projectivity. Let $(J_1, J_2) = J'$ and $(J_2, J_3) = J''$, where $J = A, B, C, X, Y, R, S$. Let $(P_1, P_1) = P_1$ and $(P_2, P_2) = P_2$, where $P = A, B, C, X, Y, R, S$. And we have the following planar perspectivities:

$$T_m A_1 B_1 X_1 \overset{T_\infty}{\doublebarwedge} T'_{ab} A'B'X' \overset{T_0}{\doublebarwedge} T_m A_2 B_2 X_2 \overset{T_\infty}{\doublebarwedge} T''_{ab} A''B''X'' \overset{T_0}{\doublebarwedge} T_m A_3 B_3 X_3,$$

$$T_m A_1 C_1 Y_1 \overset{T_\infty}{\doublebarwedge} T_{ac} A'C'Y' \overset{T_0}{\doublebarwedge} T_m A_2 C_2 Y_2 \overset{T_\infty}{\doublebarwedge} T''_{ac} A''C''Y'' \overset{T_0}{\doublebarwedge} T_m A_3 C_3 Y_3,$$

$$T_m B_1 C_1 R_1 \overset{T_\infty}{\doublebarwedge} T_{bc} B'C'R' \overset{T_0}{\doublebarwedge} T_m B_2 C_2 R_2 \text{ [illegible]},$$

$$T_m X_1 Y_1 R_1 \overset{T_\infty}{\doublebarwedge} T_{xy} X'Y'R' \overset{T_0}{\doublebarwedge} T_m X_2 Y_2 R_2,$$

$$T_m X_2 Y_2 S_2 \overset{T_\infty}{\doublebarwedge} T_{xy} X''Y''S'' \overset{T_0}{\doublebarwedge} T_m X_3 Y_3 S_3,$$

$$T_m B_2 C_2 S_2 \overset{T_\infty}{\doublebarwedge} T_{bc} B''C''S'' \overset{T_0}{\doublebarwedge} T_m B_3 C_3 S_3.$$

Let T' be the intersection of the ideal planar line with $A'A_3$, $B'''C'''$, X''', Y''', and S''' be the intersections of T_0B_2 with $T'B_3$, T_0C_2 with $T'C_3$, T_0X_2 with $T'X_3$, T_0Y_2 with $T'Y_3$, and T_0S_2 with $T'S_3$ respectively. Then

$$T_m A_2 B_2 X_2 \overset{T_0}{\doublebarwedge} T_{ab} A'B'''X''' \overset{T'}{\doublebarwedge} T_m A_3 B_3 X_3,$$

$$T_m A_2 C_2 Y_2 \overset{T_0}{\doublebarwedge} T_{ac} A'C'''Y''' \overset{T'}{\doublebarwedge} T_m A_3 C_3 Y_3,$$

$$T_m B_2 C_2 S_2 \overset{T_0}{\doublebarwedge} T_{bc} B'''C'''S''' \overset{T'}{\doublebarwedge} T_m B_3 C_3 S_3,$$

$$T_m X_2 Y_2 S_2 \overset{T_0}{\doublebarwedge} T_{xy} X'''Y'''S''' \overset{T'}{\doublebarwedge} T_m X_3 Y_3 S_3,$$ S''' being thereby determined.

Then the following triples of points are collinear:

$T_0 B' B'''$, $T_0 C' C'''$, $T_0 X' X'''$, $T_0 Y' Y'''$, $A' B' X'$, $A' C' Y'$, $A' B''' X'''$, $A' C''' Y'''$, $B' C' R'$, $X' Y' R'$, $B''' C''' S'''$, $X''' Y''' S'''$.

Let the intersection of $C'B'''$ with $Y'X'''$ be denoted H. By the Theorem of Desargues, since the planar triangles $B' C' B'''$ and $X' Y' X'''$ are perspective from A', R', T_0, and S''' are collinear, and since the planar triangles $C' B''' C'''$ and $Y' X''' Y'''$ are perspective from A', H, T_0, and S''' are collinear. Therefore R', T_0, and S''' are collinear unless H coincides with T_0, in which case B B C C are collinear and X X Y Y are collinear, and R' : S'''.

R_2 and S_2 are the projections of R' and S''' respectively upon the identical line from T_0, and therefore coincide since R', T_0, and S''' are collinear.

In the case which has been considered H cannot be ideal since R' ans S''' are ordinary points. Suppose $T_3' = T_4'$, but that $T_3'' \neq T_4''$. Then an ideal point T^r instead of the ordinary point R' would be collinear with X' and Y', and it would follow that T^r, T_0, and S''' were collinear. But this is impossible, for two ideal points cannot be collinear with an ordinary point. Hence if $T_3' = T_4'$, $T_3'' = T_4''$. An argument similar to this proves the converse. Therefore the theorem is established for Case I

Case II. A is ideal, B and C are ordinary.

The proof is very similar to that for Case I. A_1, A_2, and A_3 coincide with T_m, mhich is accordingly substituted for each of them in the perspectivities. A' and A" are replaced by ideal points T'_a and T''_a respectively. Otherwise there are practically no changes in the argument.

It is evident also that a very similar proof could be worked out if B or C had been taken as the ideal point instead of A.

Case III. If A is ordinary and B and C are ideal.

In the proof given for Case I, B_1, B_2, B_3, C_1, C_2, and C_3 coincide with T_m, which accordingly replaces each of them in the perspectivities, and B', B", C', and C" are replaced by ideal points T'_b, T''_b, T'_c, and T''_c respectively.

Cor .I. If l is an ordinary spatial line, A any ordinary spatial point not on l, and X and Y any two distinct points of the plane α determined by A and l, every point of the spatial line XY is a point of α.

Let B,C,R be the points in which l is met by AX, AY, and XY respectively. X, Y, and R are points of α by hypothesis. Let Z be any point of XY ($Z \neq X,Y,R$). A lies on YC, and Z on YR. Therefore AZ meets RC, and Z is a point of α.

12. Theorem XV. If l_1 and l_2 are any two distinct lines of an ordinary spatial plane α, l_1 and l_2 have one, and only one, point in common.

Let the plane be determined by A and l. Then let l_3 be a line through A meeting l_1 and l in B_1 and B respectively, and l_4 a line through A meeting l_2 and l in C_2 and C respectively. Let the intersections of l with l_1 and l_2 be D_1 and D_2 respectively. Since l_2 meets D_1B and D_1B_1, it also meets BB_1, say in B_2. Since l_1 meets D_2C and D_2C_2, it also meets CC_2, say in C_1. Since C_1D_1 meets AB_2 and AC_2, it meets B_2C_2, or l_1 meets l_2.

THEOREM XVI. If A, B, and C are any three non-collinear points of an ordinary spatial plane α, A, B, and C determine α.

By hypothesis at least one of the points, say C, must be ordinary. Then any ordinary point A' on AC determines with BC a plane α'. Every point of BC, and therefore every point of α' is a point of α. Let E and F be any two points of α. The line EF meets every line of α', since every line of α' is also a line of α. Since not all of the lines of α' are concurrent, CD must contain at least two points of α', and therefore must be a line of α'. Therefore $\alpha' = \alpha$, and α is determined by A, B, and C.

13. THEOREM XVII. If α is any spatial plane and l any ordinary spatial line not a line of α, l has one, and only one, point in common with α.

The truth of the theorem is obvious if α is the ideal spatial plane. It is also obvious that l cannot have two points in common with α, since it is by hypothesis not a line of α. It remains to show that l always does have one point in common with α when α is an ordinary spatial plane.

Case I. α is of Type I.

Then if $T' = T_\infty$, T is a point of α. If $T' \neq T_\infty$, and α is on N_1, π' transforms N_1 into a definite point of $\bar{l}$, say N_2. Consider three projectivities of C_m, π'_n, π''_n, and π'''_n defined as follows: π'_n is any projectivity of C_m which transforms N_1 into N_2, π''_n is that projectivity of C' which is degenerate on N_2, π'''_n is that projectivity of C'' which is degenerate on N_1.

Let $\pi''(N_2)=N_3$. Then $N=(N_1,N_2,N_3)$ is a point common to l and l_n, which is determined by $(\pi'_n,\pi''_n,\pi'''_n)$, and therefore is a point common to l and α.

The proof is practically the same when α is of I_2 or I_3.

Case II. α is of Type II_1. Let T'_a be the common first compon nent of all ideal points of α. Then if $T'=T'_a$, T is a point of α. If $T' \neq T'_a$, π' and π'_a have a common pair of homologous points, say P_1P_2. If $\pi''(P_2)=P_3$, $P=(P_1,P_2,P_3)$ is a point of l.*

The ideal line of α contains a single point, say T_1, whose second component is T''. Let $PT_1 = l_1$. Let l_2 be any ordinary line of α which does not contain T_1. If $T_1'' \neq T_2''$, π_1'' and π_2'' have a pair of homologous points, say Q_2, Q_3. Then if $\pi_2'^{-1}(Q_2)=Q_1$, $Q=(Q_1,Q_2,Q_3)$ is an ordinary point of l_2.. $T_1'=T_2'$, and π_1 and π_2 each transformP_1 into P_2. $\therefore$ $\pi_1=\pi_2$. $\pi_1'^{-1}(Q_2)=\pi_2'^{-1}(Q_2)$, and $Q= (Q_1,Q_2,Q_3)$ is on l_1 and l_2, l_1 is a line of α. P is on α, and P is on l. $\therefore$ l meets α in the point P.

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* If π'' is a degenerate projectivity which transforms P_2 into a variable P_3, then the point P is not a unique point. But in that case π_1'' must also be a projectivity of C'. $\therefore$ π_1'' and π_2'' determine a unique pair of homologous points as before.

With a few changes of subscripts and primes, this proof will hold where α is of Type II_2 or Type II_3.

Case III. α is of general type.

Then there is one, and only one, ideal point of α, say T_1, which has its second component equal to T''. (Theorem XIII)

If $T_1 = T$, the truth of the theorem is obvious. If $T_1 \neq T$, T_1 with l determines a plane α'. Let l_2 be any line of α of which T_1 is not a point. By Case I or Case II, l_2 has a unique point, say P_1, in common with α'. Let $P_1T_1 = l_1$. Then l_1 is a line common to α and α', and l_1 meets l in a unique point, say P_3, which is therefore a point common to l and α.

THEOREM XVIII. If T^β and T^γ are any two distinct ideal spatial points, T^β and T^γ uniquely determine an ideal spatial line.

Let A and A' be any two ordinary spatial points, and let B and B' be any two ordinary spatial points on the lines AT^β and $A'T^\beta$ respectively. Then B with AT^γ and B' with $A'T^\gamma$ determine two planes uniquely, say α and α' respectively. Every line of α has a point in common with α'. But α and α' have no ordinary point in common, else they would coincide. Therefore the ideal point of every line of α is a point of α'. A converse argument proves that every ideal point of α' is a point of α. Therefore T^β and T^γ uniquely determine an ideal line which is common to α and α'.

THEOREM XIX. If α and β are any two distinct spatial planes, α and β have a line in common, and no common point not on that line.

If either α or β is the ideal spatial plane, the theorem is obvious. If both α and β are ordinary, since every line of α meets β, α and β must contain at least two common points, which will uniquely determine a line common to α and β.

That no point not on that line is common to α and β follows from the fact that three non-collinear points determine a plane.

THEOREM X X. If λ_1 and λ_2 are any two distinct ideal spatial lines, λ_1 and λ_2 have one, and only one, point in common.

Let A be any ordinary spatial point. A with λ_1 and λ_2 determine two planes α_1 and α_2 respectively which have in common an ordinary spatial line through A. The ideal point of this line is the only ideal point common to α_1 and α_2, and therefore is the only point common to λ_1 and λ_2.

Cor. I. If α is an ordinary spatial plane of which the ideal line is λ, and if λ_1 is any ideal spatial line distinct from λ, λ_1 meets α in one, and only one, point.

14. We have then the following relations between the points, lines, and planes which have beend defined in terms of the points of $\bar{1}$:

A1 and A2. If A and B are any two distinct spatial points, there is one, and only one, line on both A and B. (Theorem VII, Theorem IX, Cor. IV, Theorem XI, Cor. II, and Theorem XVIII.)

A3. If l_1 and l_2 are any two distinct coplanar spatial lines, l_1 and l_2 have one point, and only one point, in common. (Theorem XIV and Theorem X X.)

C. If α is any spatial plane and l any spatial line not a line of α, l has one point, and only one point, in common with α. (Theorem XVII. and Theorem X X, Cor. I.)